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Law of the Wall for Turbulent Flows in Pressure Gradients

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Experiments show that pressure gradients have a strong effect on the law of the wall for temperature but have comparatively little effect on the law of the wall for velocity. This is contrary to the predictions of the mixing-length formulas, though these can be derived by apparently plausible dimensional analysis. To model this behavior, the choice of the transport equation for the turbulence-length scale, or its equivalent, becomes crucial; this is true both for two-equation models and for full second-moment transport models. This paper is a brief review of the experimental evidence, followed by a demonstration that the most popular "length-scale" variable, the dissipation rate ϵ, and its temperature-field equivalent ϵ_T fail to reproduce the observed behavior. A more general length-scale variable can be defined as $k^m\epsilon^n$, where k is the turbulent kinetic energy. It is shown that, overall, the ω variable (m = -1, n = 1) seems to be the best choice, with its temperature-field equivalent ω_T for heat transfer, although there is no obvious physical reason for preferring it.

Nomenclature

A^+	= Van Driest "damping constant," Eq. (28)
B^+	= damping constant for thermal boundary layer, Eq. (33)
C	= additive constant in logarithmic law, Eq. (5)
c_p	= specific heat at constant pressure
c_1, c_2	= coefficients in ϕ equation, Eq. (41)
c_μ, c_{c_1}, c_{c_2}	= coefficients in k, ϵ model
D_α, D_μ	= "damping functions," Eqs. (30) and (33)
k	= turbulent kinetic energy
l	= mixing length, Eq. (23)
m, n	= exponents in length-scale variable ϕ, Eq. (37)
p, q	= exponents in length-scale variable ϕ_T, Eq. (57)
Pr	= Prandtl number
$\dot{q}$	= total (conductive plus turbulent) heat transfer rate in y direction
T	= mean temperature
T'	= temperature fluctuation
T_τ	= friction temperature, $\dot{q}_w/(\rho c_p u_\tau)$
u, v	= mean velocity components in x, y directions
u', v'	= velocity fluctuations
u_τ	= friction velocity, $\sqrt{\tau_w/\rho}$
x, y	= coordinate directions; x streamwise, y normal to surface
α	= thermal conductivity, Eq. (6)
β	= shear-stress gradient, Eq. (20)
ϵ	= turbulent kinetic energy dissipation rate
κ	= von Karman constant, Eq. (3)
ρ	= density
$\sigma_k, \sigma_\tau, \sigma_\phi$	= turbulent "Prandtl numbers," Eqs. (38) and (39)
τ	= total (viscous plus turbulent) shear stress
ϕ	= general length-scale variable, Eq. (37)
ω	= ϵ/k

Subscripts

T	= temperature field
t	= turbulent
w	= surface ("wall")

Superscript

$+$	= "wall units"; quantity normalized by velocity scale u_τ and length scale ν/u_τ

I. Introduction: Wall-Law Derivations

THE simplest derivation of the logarithmic law for velocity in an incompressible turbulent wall flow, independent of the existence of any inner/outer "overlap" region, is attributed by Landau and Lifshitz[1] to 1944 work by Landau. We argue that, close to the wall, mean velocity u will depend only on τ_w, y, ρ, and μ, the direct effect of other quantities like u_∞ and δ being negligible once τ_w is specified.

Dimensional analysis yields

$$\frac{u}{u_\tau} = f_1\left(\frac{u_\tau y}{\nu}\right) \tag{1}$$

or equivalently

$$\frac{\partial u}{\partial y} = \frac{u_\tau}{y} f_2\left(\frac{u_\tau y}{\nu}\right) \tag{2}$$

If $u_\tau y/\nu$ is large, then viscosity should not play a significant part in the interaction between the mean flow and the larger, Reynolds-stress-carrying eddies: therefore f_2 must tend to a constant, $1/\kappa$ say. Hence, Eq. (2) becomes

$$\frac{\partial u}{\partial y} = \frac{u_\tau}{\kappa y} \tag{3}$$

or, in terms of dimensionless variables $u^+ = u/u_\tau$ and $y^+ = u_\tau y/\nu$, as

$$\frac{\partial u^+}{\partial y^+} = \frac{1}{\kappa y^+} \tag{4}$$

Integrating Eq. (4) yields the "logarithmic law"

$$u^+ = \frac{1}{\kappa} \ell n\, y^+ + C \tag{5}$$

By experiment this is valid, with $\kappa \approx 0.41$ and $C \approx 5.2$, for $y^+ > 30$ approximately (i.e., outside the "viscous wall region") but $y/\delta <$ 0.1–0.2 approximately, in boundary layers in not-too-large pressure, gradient or in pipe or duct flows, provided that the bulk Reynolds number is not too small. Below we refer to this fully turbulent part

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of the inner layer as the log-law region for simplicity, even when the validity of Eq. (5) is in dispute.

For constant-property heat transfer (small temperature differences) the additional variables are specific heat c_p, thermal conductivity α, rate of heat transfer from the wall $\dot{q}_w$, and temperature measured with respect to the wall, $T_w - T$. Then dimensional analysis gives

$$\frac{T_w - T}{\dot{q}_w/\rho c_p u_\tau} = g_1\left(\frac{u_\tau y}{\nu}, \frac{\mu c_p}{\alpha} \equiv Pr, \frac{\dot{q}_w}{\tau_w u_\tau}\right) \tag{6}$$

or equivalently

$$-\frac{\partial T}{\partial y} = \frac{\dot{q}_w/\rho c_p u_\tau}{y} g_2\left(\frac{u_\tau y}{\nu}, Pr, \frac{\dot{q}_w}{\tau_w u_\tau}\right) \tag{7}$$

Here $\dot{q}_w/\rho c_p u_\tau$ is called the *friction temperature* T_τ, analogous to the friction velocity u_τ. The last parameter on the right-hand side (RHS) of Eq. (7) represents the ratio of wall heat transfer to the internal source due to viscous dissipation. This source is small in low-speed flow, and we therefore ignore the last parameter.

If $u_\tau y/\nu$ and $(u_\tau y/\nu)Pr$ are large, both viscous and conductive effects on heat transfer should be small and g_2 should tend to a constant, $1/\kappa_T$ say, or

$$-\frac{\partial T}{\partial y} = \frac{\dot{q}_w/\rho c_p u_\tau}{\kappa_T y} = \frac{T_\tau}{\kappa_T y} \tag{8}$$

By defining a new dimensionless temperature $T^+ \equiv (T_w - T)/T_\tau$ Eq. (8) can be written in terms of dimensionless variables as

$$\frac{\partial T^+}{\partial y^+} = \frac{1}{\kappa_T y^+} = \frac{Pr_t}{\kappa y^+} \tag{9}$$

which integrates to give

$$T^+ = \frac{1}{\kappa_T}\ell n\, y^+ + C_T \tag{10}$$

where the *intercept* C_T is a function of Pr and $C_T \approx 3.9$ for air ($Pr = 0.71$). The turbulent Prandtl number Pr_t is defined in terms of *local* variables by

$$Pr_t = \frac{\overline{u'v'}/(\partial u/\partial y)}{\overline{v'T'}/(\partial T/\partial y)} = -\frac{\overline{u'v'}^+\partial T^+/\partial y^+}{\overline{v'T'}^+\partial u^+/\partial y^+} \tag{11}$$

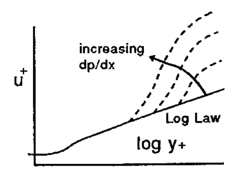
where $-\overline{u'v'}^+$ and $\overline{v'T'}^+$ are dimensionless turbulent shear stress and heat fluxes, $-\overline{u'v'}/u_\tau^2$ and $\overline{v'T'}/(u_\tau T_\tau)$, respectively. By experiment,[2] Eq. (9) is valid for $y^+ > 30$ in pipe or duct flows and in boundary layers with zero pressure gradient (ZPG), with $\kappa_T \approx 0.48$—a little larger than κ, implying that the turbulent Prandtl number is approximately 0.85 in the log-law region.

Townsend[3] used the transport equation for turbulent kinetic energy (TKE) to justify generalizing the law of the wall for ZPG flow to cases where shear stress varies with y. If one ignores turbulent diffusion of TKE, which Townsend's estimates suggest is small except near separation, then the analysis justifies replacing u_τ in Eq. (3) by local $(\tau/\rho)^{\frac{1}{2}}$, which yields

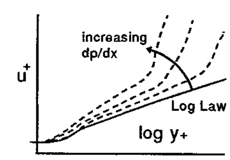
$$\frac{\partial u}{\partial y} = \frac{\sqrt{\tau/\rho}}{\kappa y} \tag{12}$$

This is the classical mixing-length formula but has apparently better justification. For some special cases the equation can be integrated analytically, and for a linear shear-stress profile it yields Eq. (5) plus a function of $(y/\tau_w)\, \partial\tau/\partial y$. Note that although the von Kármán constant κ is supposed to be independent of $\partial\tau/\partial y$, the constant of integration that replaces C in Eq. (5) can in principle depend on the value of $\partial\tau^+/\partial y^+$, where $\tau^+ \equiv \tau/\tau_w$, in the viscous wall region. Townsend did not discuss heat transfer, but analogous arguments for the temperature field justify replacing u_τ in Eq. (8), which yields

$$-\frac{\partial T}{\partial y} = \frac{\dot{q}_w/\rho c_p\sqrt{\tau/\rho}}{\kappa_T y} \tag{13}$$



a)



b)

Fig. 1 Possible modes of velocity profile departing from log law in APGs: a) progressive departure and b) general departure (reproduced from Galbraith et al.[6]).

In terms of *wall dimensionless variables*, Eqs. (12) and (13) can be written as

$$\frac{\partial u^+}{\partial y^+} = \frac{\sqrt{\tau^+}}{\kappa y^+} \tag{14}$$

$$\frac{\partial T^+}{\partial y^+} = \frac{1}{\kappa_T y^+\sqrt{\tau^+}} = \frac{Pr_t}{\kappa y^+\sqrt{\tau^+}} \tag{15}$$

There is a fair amount of support for the *extended law of the wall* using local τ/ρ, both in flows with distributed suction or injection, where τ changes with y, and in compressible flow, where ρ and $\dot{q}$ also change with y. Nevertheless, as is well known, the logarithmic law of the wall for velocity [Eq. (5)] is still followed in adverse pressure gradients (APGs), nearly up to separation. At low Reynolds numbers the changes in total shear stress through the viscous wall region, $0 < y^+ < 30$, can be large, leading to a decrease in C in APG, and conversely.[4] The von Kármán constant, κ, on the other hand, seems to remain constant except at extremely low Reynolds numbers. For a more detailed discussion of the limits of validity of the law of the wall see Bradshaw and Huang.[5] With the exception of the low-Reynolds-number behavior the departure of the velocity profiles from the log law is "progressive" rather than "general," in the terminology of Galbraith et al.[6] (see Fig. 1). Townsend's formula for velocity [Eq. (14)], showing that the slope of u^+ is proportional to $(\tau^+)^{\frac{1}{2}}$, implies a general departure. In contrast, for the temperature law of the wall, the measurements of Thielbahr et al.[7] and Blackwell et al.,[8] shown in Figs. 13–11 of Kays and Crawford,[2] display a much larger slope in favorable pressure gradient and a much smaller slope in APG. Now τ increases with y in APG, and conversely, whereas the profile of $\dot{q}$ is little affected by pressure gradient. Therefore the trend of temperature gradient given in Eq. (15) is in the right direction, but as will be seen later, it is far too small to explain the observed changes in T^+.

In this paper we assume as a working approximation that the log law for velocity [Eq. (5)] is universal and discuss the constraints that a turbulence model must satisfy in order to reproduce the log law and also the observed nonuniversal behavior of the temperature profile. The discussion centers on the transport equation for a length scale or its equivalent, and although we consider two-equation *eddy-viscosity* models for simplicity, the qualitative conclusions also apply to transport-equation models. Section II sets out the modifications to the formulas for eddy viscosity and turbulent Prandtl number that are required to reproduce the observed results. Sections III and IV discuss the optimum form of *length-scale* transport equation for a two-equation eddy-viscosity model, both for the velocity field and for the temperature field. The conclusion is that the optimum length-scale equation is that for $\omega \equiv \epsilon/k$: This differs only trivially from the variable used in the κ–ω model of Wilcox.[9] The optimum two-equation heat transfer model, which evaluates the eddy conductivity directly without using a turbulent Prandtl number, appears to be that based on ω_T, the ratio of the rate of dissipation of mean-square temperature fluctuation to the mean-square temperature fluctuation itself.

II. Turbulent Viscosity and Turbulent Prandtl Number

If we define μ_t^+ and α_t^+, the dimensionless turbulent viscosity and conductivity, as

$$\mu_t^+ = \frac{\mu_t}{\mu} = \frac{\overline{u'v'}/u_\tau^2}{\partial u^+/\partial y^+} \tag{16}$$

$$\alpha_t^+ = \frac{\alpha_t}{c_p\mu} = \frac{\overline{v'T'}/u_\tau T_\tau}{\partial T^+/\partial y^+} \tag{17}$$

then the total (viscous plus turbulent) shear stress and the total heat-flux rate are given, in dimensionless form, by

$$(1+\mu_t^+)\frac{\partial u^+}{\partial y^+} = \tau^+ \tag{18}$$

$$(1+\alpha_t^+)\frac{\partial T^+}{\partial y^+} = \dot{q}^+ \tag{19}$$

where τ^+ and $\dot{q}^+$ are dimensionless shear stress and heat flux. In the inner layer, $0 < y/\delta < 0.1$–0.2,

$$\tau^+ \approx 1 + \beta y^+ \tag{20}$$

and

$$\dot{q}^+ \approx 1 \tag{21}$$

are good approximations for a constant-temperature wall in pressure gradient (pressure gradient does not affect heat flux directly, and β is not necessarily closely related to the pressure gradient except at $y = 0$ because the acceleration may be significant).

To satisfy Eq. (4) in the log-law region, Eq. (18) shows that u_t^+ must be proportional to $y^+\tau^+$:

$$\mu_t^+ = \kappa y^+\tau^+ \qquad \text{or} \qquad \mu_t = \kappa y\tau/u_\tau \tag{22}$$

This formula implies that the mixing length should vary as

$$l^+ = \kappa y^+\sqrt{\tau^+} \qquad \text{or} \qquad l = \kappa y\sqrt{\tau/\tau_w} \tag{23}$$

instead of

$$l^+ = \kappa y^+ \qquad \text{or} \qquad l = \kappa y \tag{24}$$

which leads to

$$\mu_t^+ = \kappa y^+\sqrt{\tau^+} \qquad \text{or} \qquad \mu_t = \rho\kappa y\sqrt{\tau/\rho} \tag{25}$$

as originally proposed by Townsend[3] and used in Eqs. (14) and (15). Equation (23) yields the velocity and temperature formulas:

$$\frac{\partial u^+}{\partial y^+} = \frac{1}{\kappa y^+} \tag{26}$$

$$\frac{\partial T^+}{\partial y^+} = \frac{Pr_t}{\kappa y^+\tau^+} \tag{27}$$

We shall see later in this section that $Pr_t \approx 0.85$ even in flows with pressure gradients, though the arguments that led to $Pr_t = \text{const} = \kappa/\kappa_T$ [see Eq. (10)] are apparently no longer valid.

To obtain the correct intercepts in the log laws for velocity and temperature, C and C_T in Eqs. (5) and (10), for ZPG flow, one may propose a variety of *damping functions* in μ_t^+ and/or α_t^+ in the viscous/conductive wall region. One of the most popular is by Van Driest,[10] who employed an exponential damping on the mixing length:

$$l^+ = \kappa y^+\left[1 - \exp\left(-\frac{y^+}{A^+}\right)\right] \tag{28}$$

(the exponential is best regarded as wholly empirical: Other analytic functions have been suggested). This can easily be reformulated as a damping function for μ_t;

$$\mu_t^+ = \kappa y^+ D_\mu \tag{29}$$

where

$$D_\mu = \left[1 - \exp\left(-\frac{y^+}{A^+}\right)\right]^2 \kappa y^+ \frac{\partial u^+}{\partial y^+} \tag{30}$$

The value of A^+ is 25.5, chosen to match the law of the wall for velocity [Eq. (5)] at large y^+. It should be noted that the Van Driest formula gives $\overline{u'v'} \sim y^4$ as $y \to 0$ whereas the true behavior is $\overline{u'v'} \sim y^3$: This is not an important issue here, as any error is absorbed in A^+. It should also be noted that integrating Eq. (19) with a constant turbulent Prandtl number equal to 0.85, and μ_t from the Van Driest formula, leads to a smaller intercept than the experimental value, $C_T = 3.9$. To increase C_T to the experimental value, Kays and Crawford[2] suggested that Pr_t should be increased near the wall and proposed the following formula:

$$Pr_t = \frac{1}{0.5882 + 0.231\mu_t^+ - 0.0454\mu_t^{+2}[-\exp(-5.092/\mu_t^+)]} \tag{31}$$

As seen in Fig. 2, Eq. (31) increases the value of Pr_t near the wall and asymptotes to $Pr_t \approx 1.7$ on the surface. Although the marked increase of turbulent Prandtl number near the wall proposed by Kays and Crawford is supported by experiments, it is not supported by the direct numerical simulations of Kim[11] and Bell and Ferziger,[12] who found that Pr_t is about 1.1 at the wall. Kays[13] argued that the difference may be caused by the low Reynolds number of the DNS results. Alternatively,[14] one can use a damping function for α_t similar to that used for μ_t:

$$\alpha_t^+ = \kappa_T\, y^+\, D_\alpha \tag{32}$$

where

$$D_\alpha = \left[1 - \exp\left(-\frac{y^+}{B^+}\right)\right]^2 \kappa y^+ \frac{\partial u^+}{\partial y^+} \tag{33}$$

The value of B^+ is assigned to be 33 in order to match the law of the wall for temperature [Eq. (10)] at large y^+. The resulting turbulent Prandtl number obtained using Eq. (32) is shown in Fig. 2. It should be noted that, if one assigns $B^+ = A^+$ in Eq. (33), a constant Prandtl number is implied. As can be seen from Fig. 2, the predicted value of C_T is not sensitive to the magnitude of Pr_t very near the wall because in this region turbulent transport is nearly zero: Thus the fact that the "damping function" model predicts the right C_T with only a small rise in Pr_t does not prove that the DNS results are right and the Kays–Crawford data correlation is wrong. The main virtue of the damping function model is that it is easily adjustable: We will use it later in analyzing the law of the wall for temperature in flows with pressure gradient, where the experimental data show that C_T as well as κ_T is strongly affected by the pressure gradient. It should be noted that both the experimental data and the DNS results show that Pr_t tends toward a nearly constant value of about 0.7 in the outer part of a boundary layer ($y/\delta > 0.2$).

Attention is now focused on the law of the wall for temperature in the presence of pressure gradients. Figure 3 shows comparisons with the data of Blackwell et al.[8] (case 110871, corresponding to the strongest APG in their experiments) by integrating Eqs. (18) and (19) with values of τ^+ and $\dot{q}^+$ taken directly from experimental data. The values for μ_t^+ are obtained based on the original mixing length and the modified formulas, Eqs. (25) and (22), respectively. The Van Driest damping function tuned for flows in ZPG [Eq. (30)] is applied in both equations:

$$\mu_t^+ = \begin{cases} \kappa y^+\sqrt{\tau^+}D_\mu, & \text{original mixing-length model} \\ \kappa y^+\tau^+ D_\mu, & \text{modified mixing-length model} \end{cases} \tag{34}$$

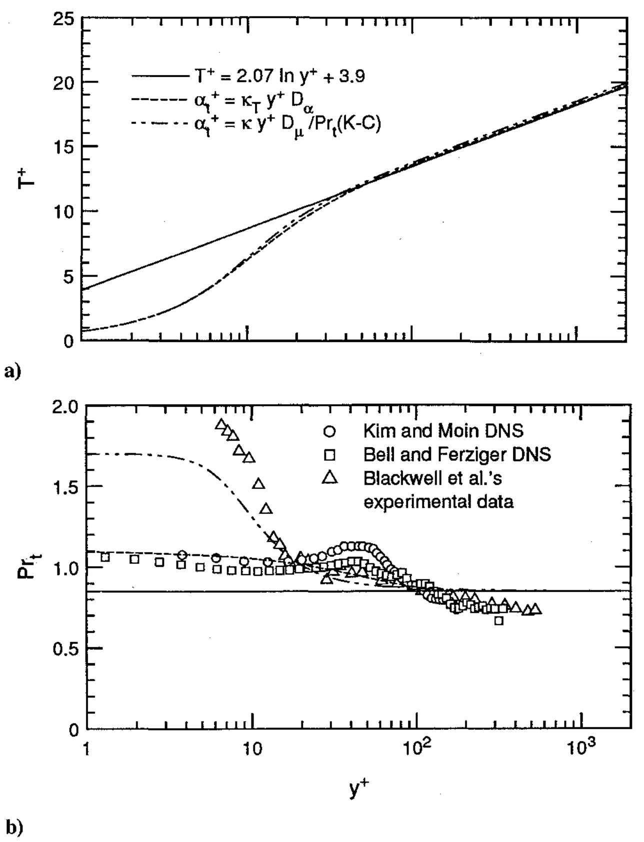


Fig. 2 Prediction of law of wall for temperature in ZPG using Kays-Crawford (KC) formula for Pr_t and a Van-Driest-type "damping function" D_α for α_t^+.

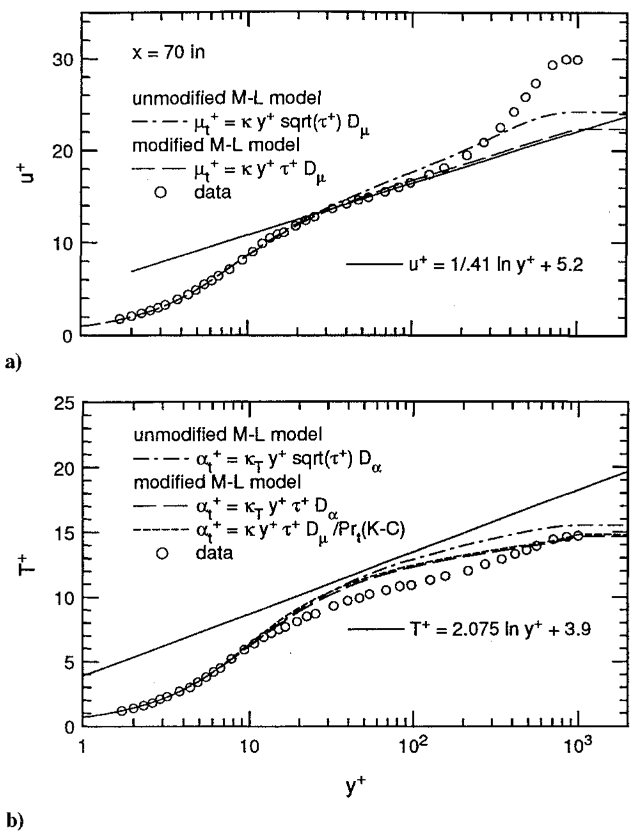


Fig. 3 Prediction of law of wall for velocity and temperature in APG. Data of Blackwell et al.[8] ML stands for the mixing-length model.

The turbulent conductivity is evaluated according to the following three forms of α_t^+:

$$\alpha_t^+ = \frac{1}{Pr_t}\kappa y^+ D_\alpha = \begin{cases} \kappa_T y^+ \sqrt{\tau^+} D_\alpha, & \text{original mixing-length model} \\ \kappa_T y^+ \tau^+ D_\alpha, & \text{modified mixing-length model} \\ \kappa y^+ \tau^+ D_\mu / Pr_t \text{ [Eq. (31)]}, & \text{modified mixing-length model} \end{cases} \tag{35}$$

where the first two cases use D_α as defined in Eq. (33) whereas the last case uses the turbulent Prandtl number definition given by Kays and Crawford [Eq. (31)]. [Equations (31) and (33) are both calibrated to match the law of the wall for temperature in ZPG]. As can be seen from Fig. 3a, the unmodified mixing-length formula gives rise to a higher slope in u^+ (smaller κ) than experiment. Kays and Crawford[2] and Volino and Simon[15] remedy this problem by adjusting the value of A^+ in Eq. (28) according to the strength of the pressure gradient. However, since the rise in the slope is associated with the deficiency of the model in the log-law region, as shown in Eq. (14), the adjustment of the damping function in the viscous wall region to compensate for the change of slope caused by misbehavior of the model in the logarithm layer seems inappropriate.

Figure 3b shows the comparison of the temperature profiles. As may be seen from the figure, the experimental data not only show a larger value of κ_T (or a smaller slope in T^+) but also have a smaller value of the intercept C_T than those for the ZPG case. In contrast, all three models predict too large a value of intercept C_T, suggesting that the turbulent Prandtl number near the wall is overpredicted. The average slope in the logarithm region ($30 < y^+ < 200$) for the modified mixing-length model seems to match the experimental data fairly well, suggesting that $Pr_t \approx 0.85$ is an adequate assumption. Indeed, when applying the same analysis to Thielbahr's favorable pressure gradient data,[7] a constant value of Pr_t equal to 0.85 also seems to be quite adequate in the log-law region, although here, in contrast to the case of APG, the constant C_T is larger than in ZPG, indicating that an increase of Pr_t near the surface is needed. In general C_T, like C, must be a function of $(\partial\tau^+/\partial y^+)_w$.

To get a more realistic comparison with the temperature profile in APG, we adjusted the damping constant B^+ in Eq. (33) from 33 to 22 so as to reduce C_T. Figure 4a shows that the modified mixing-length model now matched the data very well in the log-law region. As can be seen from Fig. 4b, the turbulent Prandtl number near the wall is reduced for APG cases. The drop in Pr_t near $y^+ \approx 10$ seems to be supported by the experimental data. For $y^+ < 10$, the experimental data show an overshoot of Pr_t toward the wall, but once again, the detailed variation in turbulent Prandtl number very near the wall is not very important to the predicted value of C_T. Moreover, the experimental data in the log-law region seem to show that Pr_t is slightly lower in APG than in ZPG. The Reynolds number in Blackwell's experiments is small ($Re_\theta \approx 3600$), so this behavior may be attributable to the influence of the outer layer rather than the pressure gradient as such.

III. Modeling Requirements

For simplicity we discuss two-equation models, in which the dimensionless turbulent viscosity μ_t^+ is expressed in terms of two independent dimensionless variables, e.g., turbulence kinetic energy and its dissipation rate, k^+ and ϵ^+:

$$\mu_t^+ \equiv \mu_t/\mu = c_\mu k^{+2}/\epsilon^+ \tag{36}$$

where c_μ is fixed at 0.09 to satisfy $-\overline{u'v'}/k \approx 0.3$ in local energy equilibrium (production = dissipation). To express a general form for the length-scale equation, we define a general variable ϕ^+ in terms of k^+ and ϵ^+ as

$$\phi^+ = (k^+)^m(\epsilon^+)^n \tag{37}$$

For example $m = 1$ and $n = 1$ gives $\phi \equiv \omega = \epsilon/k$.

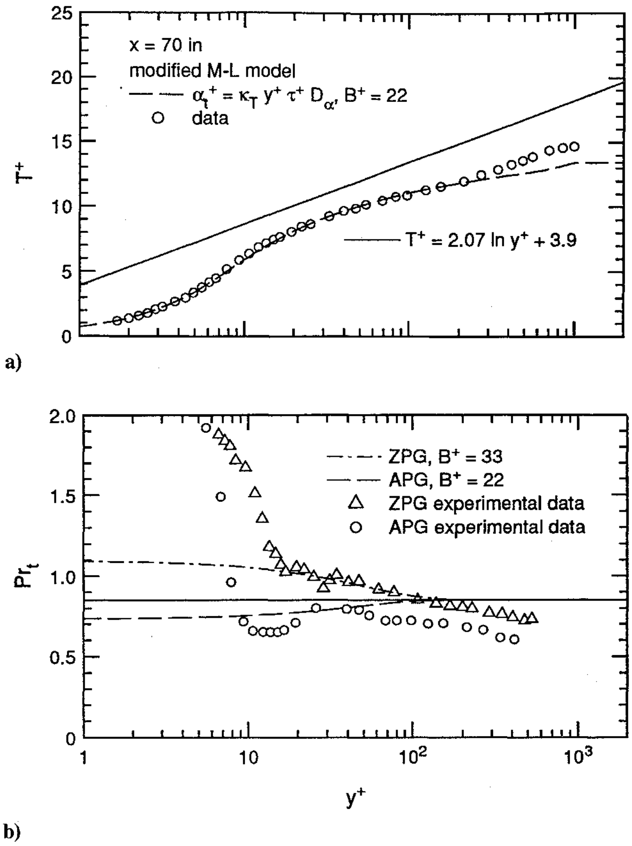


Fig. 4 Prediction of law of wall for temperature in APG using Eq. (32).

The standard transport equations for k^+ and ϕ^+, in thin-shear-layer form, are given as

$$u^+\frac{\partial k^+}{\partial x^+} + v^+\frac{\partial k^+}{\partial y^+} = \frac{\partial}{\partial y^+}\left[\left(1+\frac{\mu_t^+}{\sigma_k}\right)\frac{\partial k^+}{\partial y^+}\right] + P_{k^+} - \epsilon^+ \quad (38)$$

$$u^+\frac{\partial \phi^+}{\partial x^+} + v^+\frac{\partial \phi^+}{\partial y^+} = \frac{\partial}{\partial y^+}\left[\left(1+\frac{\mu_t^+}{\sigma_\phi}\right)\frac{\partial \phi^+}{\partial y^+}\right] + c_1 P_{k^+}\frac{\phi^+}{k^+} - c_2\epsilon^+\frac{\phi^+}{k^+} \quad (39)$$

where σ_k and σ_ϕ are turbulent *Prandtl numbers* for k and ϕ, respectively; P_{k^+} is the turbulent-kinetic-energy production

$$P_{k^+} = -\frac{\overline{u'v'}}{u_\tau^2}\frac{\partial u^+}{\partial y^+} = \mu_t^+\left(\frac{\partial u^+}{\partial y^+}\right)^2 \quad (40)$$

and c_1 and c_2 are dimensionless coefficients, related to the corresponding coefficients of the ϵ equation, $c_{\epsilon,1}$ and $c_{\epsilon,2}$, by

$$c_1 = nc_{\epsilon,1} + m, \qquad c_2 = nc_{\epsilon,2} + m \quad (41)$$

In order to satisfy the law of the wall [Eq. (5)], for ZPG flow, a relationship among the model coefficients must be established:

$$\frac{c_\mu^{\frac{1}{2}}\sigma_\phi}{n^2\kappa^2}(c_2 - c_1) = 1 \quad (42)$$

Equation (42) is obtained from the ϕ^+ equation [Eq. (39)] with the following assumptions: 1) the flow is in local equilibrium, leading to $-\overline{u'v'}/k = \tau_w/\rho k = \sqrt{c_\mu}$; 2) the convection of ϕ^+ is negligible; and 3) $\partial u^+/\partial y^+ = 1/\kappa y^+$. For a flow in pressure gradient where $\partial k/\partial y$ is significant even in the log-law region, the first assumption requires that σ_k is infinite, and the local equilibrium assumption then leads to $\tau^+/k^+ = \sqrt{c_\mu}$, a relation supported experimentally in ZPG flows although, in APG, τ^+/k^+ tends to decrease[16]; the second assumption is retained; and the third assumption is replaced by $\partial u^+/\partial y^+ = 1/(\kappa^* y^+)$, where κ^* is a free parameter to be determined. The relation corresponding to Eq. (42) is

$$\frac{c_\mu^{\frac{1}{2}}\sigma_\phi}{n^2\kappa^{*2}}(c_2 - c_1) = \tau^+ + \frac{m-2n^2-2mn}{n^2}y^+\frac{\partial\tau^+}{\partial y^+} + \frac{m+n}{n^2}y^{+2}\frac{\partial^2\tau^+}{\partial y^{+2}} + \left(\frac{m+n}{n}\right)^2\frac{y^{+2}}{\tau^+}\left(\frac{\partial\tau^+}{\partial y^+}\right)^2 \quad (43)$$

In the log-law region, if we substitute the linear shear-stress profile $\tau^+ = 1 + \beta y^+$ into Eq. (43), we get

$$\frac{c_\mu^{\frac{1}{2}}\sigma_\phi}{n^2\kappa^{*2}}(c_2 - c_1) = \tau^+ + \frac{m-2n^2-2mn}{n^2}y^+\beta + \left(\frac{m+n}{n}\right)^2\frac{y^{+2}}{\tau^+}\beta^2 \quad (44)$$

For the k–ϵ model, $m = 0$ and $n = 1$, and Eq. (44) becomes

$$\frac{c_\mu^{\frac{1}{2}}\sigma_\phi}{\kappa^{*2}}(c_2 - c_1) = \tau^+ - 2y^+\beta + \frac{y^{+2}\beta^2}{\tau^+} = \frac{(\tau^+ - \beta y^+)^2}{\tau^+} = \frac{1}{\tau^+} \quad (45)$$

An analogous but more complicated formula was derived by Rodi and Scheuerer[17]; their analysis involves a slightly inconsistent approximation for the eddy viscosity, which leads to significant differences in the algebra. To first-order approximation, it follows that κ^* must be made equal to $\kappa\sqrt{\tau^+}$ in order to keep the model constants the same as in ZPG flow. Equation (45) thus replaces Eq. (26), which yields the log law, by

$$\frac{\partial u^+}{\partial y^+} \approx \frac{1}{\kappa y^+\sqrt{\tau^+}} \quad (46)$$

Since τ^+ increases with increasing APG, $\partial u^+/\partial y^+$ is predicted to decrease correspondingly. This is, of course, inconsistent with experimental evidence [Eq. (26)]. As a result, failure of the k–ϵ model in predicting flows with pressure gradients is anticipated. The mixing-length model [Eq. (14)] also gives rise to an incorrect law of the wall for velocity, but the effect of APG on the k–ϵ and mixing-length models is opposite: They respectively predict decreasing and increasing $\partial u^+/\partial y^+$ at given y^+.

If the general two-equation (k–ϕ) model is to satisfy Eq. (26) for non-ZPG Eq. (44) shows that we require

$$n = -m \quad (47)$$

and

$$\frac{m-2n^2-2mn}{n^2} = -1 \quad (48)$$

so that

$$\frac{c_\mu^{\frac{1}{2}}\sigma_\phi}{n^2\kappa^2}(c_2 - c_1) = \tau^+ - y^+\beta = 1 \quad (49)$$

The solution of Eqs. (47) and (48) is $m = -1$ and $n = 1$, and the length-scale variable is $\omega\,(\equiv \epsilon/k)$. The k–ω model does not necessarily guarantee a solution that is independent of pressure gradient, although it can satisfy the law of the wall exactly if σ_k is infinitely large. The analysis leading to Eq. (49) assumed local equilibrium for k, but the diffusion of k is not zero in Eq. (38) in the presence of pressure gradients (strictly, nonzero $\partial k/\partial y$), which may increase the deviation of the velocity profile from the law of the wall [Eq. (26)]. To see the combined effects resulting from both the diffusion of k and the length-scale equations on the log-law velocity profile, a perturbation method is applied to the governing Eqs. (18), (38) and (39) in the Couette-flow case (neglecting the convection terms). The first-order solution yields

$$\frac{\partial u^+}{\partial y^+} = \frac{1}{\kappa y^+} + \beta\{\kappa^4 n(n-1)(n-2) + \kappa^2\sqrt{c_\mu}\sigma_\phi \times[(n-1)c_1 - (n+1)c_2] + 2\kappa^2\sqrt{c_\mu}\sigma_k \times[m(1-2n) + n^2(n+m-1)] + 2c_\mu\sigma_\phi\sigma_k \times(m+n)(c_1-c_2)\}/\{\kappa[-n(n-1)(n-2)\kappa^2 \times + \sqrt{c_\mu}\sigma_\phi(c_2-c_1)(n+1)](2\sqrt{c_\mu}\sigma_k - \kappa^2)\} + \mathcal{O}[\beta^2]$$

$$= \frac{1}{\kappa y^+} - \begin{cases} \beta\left(\sqrt{c_\mu}\sigma_k(c_2-c_1) + \kappa^2 c_2\right)/\left[\kappa(c_2-c_1)\left(2\sqrt{c_\mu}\sigma_k - \kappa^2\right)\right] + \mathcal{O}[\beta^2], & k\text{–}\epsilon \text{ model} \\ \beta\kappa c_2/\left[(c_2-c_1)\left(2\sqrt{c_\mu}\sigma_k - \kappa^2\right)\right] + \mathcal{O}[\beta^2], & k\text{–}\omega \text{ model} \end{cases} \quad (50)$$

By substituting the standard model constants into Eq. (50), it becomes

$$\frac{\partial u^+}{\partial y^+} = \frac{1}{\kappa y^+} - \begin{cases} 5.57\beta + \mathcal{O}[\beta^2], & k\text{–}\epsilon \text{ model} \\ 1.19\beta + \mathcal{O}[\beta^2], & k\text{–}\omega \text{ model} \end{cases} \quad (51)$$

The value of β in Blackwell's experiment, shown in Fig. 3, is approximately 0.0081 and the logarithm layer in the vicinity of $y^+ = 50$ does not seem to be affected by the wake. By substituting the above conditions into Eq. (51), the values of the dimensionless velocity slope $\partial u^+/\partial y^+$ at $y^+ = 50$ are 0.039 for the k–ω model and 0.0036 (a much smaller value than that of the k–ω model) for the standard k–ϵ model, as opposed to the standard log-law value of 0.049.

If one sets σ_k to infinity, the local equilibrium assumption is enforced and Eq. (50) becomes

$$\frac{\partial u^+}{\partial y^+} = \frac{1}{\kappa y^+} - \begin{cases} \beta/(2k) + \mathcal{O}[\beta^2], & k\text{–}\epsilon \text{ model} \\ 0 + \mathcal{O}[\beta^2], & k\text{–}\omega \text{ model} \end{cases} \quad (52)$$

This equation can be shown to be identical to Eq. (46) to first order. If one expands Eq. (46) in powers of β, the equation becomes

$$\frac{\partial u^+}{\partial y^+} \approx \frac{1}{\kappa y^+\sqrt{\tau^+}} = \frac{1}{\kappa y^+\sqrt{1+\beta y^+}} \approx \frac{1}{\kappa y^+} - \frac{\beta}{2\kappa} + \mathcal{O}[\beta^2] \quad (53)$$

which is identical to the k–ϵ solution shown in Eq. (52).

Figure 5 shows predictions of Blackwell's data using the Wilcox k–ω model and the low-Reynolds-number version of the k–ϵ model by Coakley and Huang[18] [due to a slightly different set of model constants used, the coefficient associated with the β term in Eq. (51) is 5.86]. As seen from Fig. 5a, the k–ϵ model predicts too large a value of κ (or too small a value of $\partial u^+/\partial y^+$) for y^+ between 30 and 200, leading to too small a value of u_∞/u_τ ($\equiv\sqrt{2/c_f}$) at the edge of the boundary layer, whereas the k–ω model enables relatively good agreement with the experimental data. Thus, the skin friction is overpredicted by the k–ϵ model. It should be noted that this misbehavior is mainly associated with the log-law region and hence has little to do with the choice of the *low-Reynolds-number* terms, which are active only in the viscous wall region. Rodi and Scheuerer[17] and Wilcox[19] used different variants of the k–ϵ model in various pressure gradient flows and all predicted similar incorrect behavior.

Figure 5b compares temperature profiles. Both predictions assume a constant value of Pr_t equal to 0.85. The predictions of temperature in the viscous wall region are close to the experimental data. This, however, is fortuitous because, as discussed in the previous section, assuming constant turbulent Prandtl number, $Pr_t = 0.85$, leads to a lower intercept than that of the standard law of the wall.

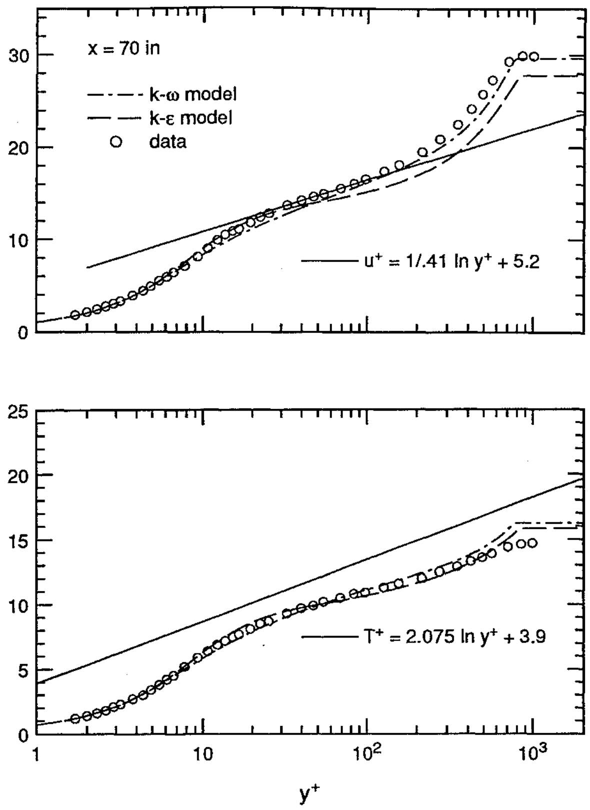


Fig. 5 Prediction of law of wall for velocity and temperature in APG using k–ϵ and k–ω models (assuming constant Pr_t equal to 0.85).

Moreover, the prediction of the k–ϵ model seems to show a smaller value of $\partial T^+/\partial y^+$ for y^+ between 30 and 200. Because the turbulent Prandtl number is assumed to be a constant, the underprediction of $\partial u^+/\partial y^+$ by the k–ϵ model is mainly responsible for the resulting dimensionless temperature slope $\partial T^+/\partial y^+$ being smaller than the experimental value, as seen from Eq. (11).

Another interesting property of the k–ω model is worthy of mention: It requires no low-Reynolds-number terms. By choosing specific values for the constants in the ϵ equation, say $c_{\epsilon,1} = 1.56$, $c_{\epsilon,2} = 1.83$, and $c_\mu = 0.09$ (as recommended in the standard k–ω model), using Eq. (41), and evaluating σ_ϕ from Eq. (42), one may integrate the Couette-flow form of the governing Eqs. (18), (38), and (39) from the wall to a position outside the viscous wall region. The following boundary conditions are prescribed at the wall:

$$y^+ = 0: \qquad u^+ = 0, \qquad k^+ = 0, \qquad \frac{\partial k^+}{\partial y^+} = 0 \quad (54)$$

It should be noted that a boundary condition for ϕ is not required since its value will be determined as part of the solution so as to satisfy $\partial k^+/\partial y^+ = 0$ at the wall. The solution to the Couette-flow equations gives rise to a family of curves satisfying

$$u^+ = \frac{1}{\kappa}\ln y^+ + C(m, n) \quad (55)$$

at large y^+. The value of the intercept constant C depends on the choice of m and n and is found numerically to be not very sensitive to the choice of σ_k. Figure 6 shows the curve for various combinations of m and n that yield $C \approx 5.2$. As can be seen from the figure, the ω variable, $m = -1$, $n = 1$, lies very close to the curve.

IV. Heat Transfer Modeling

To improve on the assumption of prescribed turbulent Prandtl number, we need to model turbulent conductivity α_t directly. Again, consider a two-equation model formulation in which dimensionless

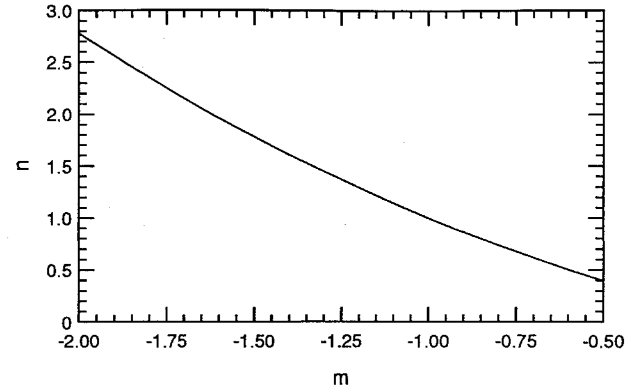


Fig. 6 The $k^m\epsilon^n$ models requiring no low-Reynolds-number terms.

turbulent conductivity α_t^+ can be assumed to have the following form[20]:

$$\alpha_t^+ \equiv \frac{\alpha_t}{c_p\mu} = c_\alpha k^+\left(\frac{k^+}{\epsilon^+}\right)^{\frac{1}{2}}\left(\frac{k_T^+}{\epsilon_T^+}\right)^{\frac{1}{2}} \tag{56}$$

where $k_T^+ \equiv \frac{1}{2}(\overline{T'^2}/T_\tau^2)$, ϵ_T^+ is the dimensionless dissipation rate of k_T^+, and c_α is a constant to be determined later. By analogy with the discussion in the previous section, a general length-scale variable ϕ_T^+ is defined in terms of k_T^+ an ϵ_T^+:

$$\phi_T^+ = (k_T^+)^p(\epsilon_T^+)^q \tag{57}$$

Equation (57) allows construction of a variety of length-scale variables by assigning various values to p and q: For example, $p = -1$ and $q = 1$ gives $\phi_T \equiv \omega_T = \epsilon_T/k_T$.

The standard transport equations for k_T^+ and ϕ_T^+ in thin-shear-layer form are usually given as follows[21–23]:

$$u^+\frac{\partial k_T^+}{\partial x^+} + v^+\frac{\partial k_T^+}{\partial y^+} = \frac{\partial}{\partial y^+}\left[\left(\frac{1}{Pr}+\frac{\alpha_t^+}{\sigma_{k_T}}\right)\frac{\partial k_T^+}{\partial y^+}\right] + P_{k_T^+} - \epsilon_T^+ \tag{58}$$

$$u^+\frac{\partial \phi_T^+}{\partial x^+} + v^+\frac{\partial \phi_T^+}{\partial y^+} = \frac{\partial}{\partial y^+}\left[\left(\frac{1}{Pr}+\frac{\alpha_t^+}{\sigma_{\phi_T}}\right)\frac{\partial \phi_T^+}{\partial y^+}\right]$$
$$+ c_{P_1}P_{k_T^+}\frac{\phi_T^+}{k_T^+} + c_{P_2}P_{k^+}\frac{\phi_T^+}{k^+} - c_{D_1}\epsilon_T^+\frac{\phi_T^+}{k_T^+} - c_{D_2}\epsilon^+\frac{\phi_T^+}{k^+} \tag{59}$$

where σ_{k_T} and σ_{ϕ_T} are turbulent Prandtl numbers for k_T and ϕ_T, respectively; P_{k^+} is the turbulent kinetic energy production defined by Eq. (40) and $P_{k_T^+}$ is the production arising from mean temperature gradients,

$$P_{k_T^+} = \frac{\overline{v'T'}}{u_\tau T_\tau}\frac{\partial T^+}{\partial y^+} = \alpha_t^+\left(\frac{\partial T^+}{\partial y^+}\right)^2 \tag{60}$$

and c_{P_1}, c_{P_2}, c_{D_1}, and c_{D_2} are dimensionless coefficients, related to the corresponding coefficients of the ϵ_T equation, c_{ϵ,P_1}, c_{ϵ,P_2}, c_{ϵ,D_1}, and c_{ϵ,D_2} by

$$c_{P_1} = qc_{\epsilon,P_1} + p, \qquad c_{P_2} = qc_{\epsilon,P_2}$$
$$c_{D_1} = qc_{\epsilon,D_1} + p, \qquad c_{D_2} = qc_{\epsilon,D_2} \tag{61}$$

To analyze the thermal field in the logarithmic region, we assume that the velocity field and turbulent quantities satisfy the law of the wall exactly; i.e., the velocity satisfies Eq. (4) and the turbulence is in local equilibrium. The following assumptions are also made: 1) the temperature variance k_T is in local equilibrium, i.e., $P_{k_T^+} = \epsilon_T^+$, which leads to $k_T^+ = (c_\mu R)^{\frac{1}{2}}/(c_\alpha\tau^+)$, where R is the ratio of scalar to velocity time scale $R \equiv (k_T/\epsilon_T)/(k/\epsilon)$; 2) the convection of ϕ_T^+ is negligible; and 3) $\partial T^+/\partial y^+ = 1/(\tau^+\kappa_T^* y^+)$, where κ_T^* is again a free parameter to be determined. By substituting the above assumptions into the ϕ_T^+ equation, the following relation, analogous to Eq. (43), is obtained:

$$\left[\left(c_{D_1}-c_{P_1}\right)\frac{1}{R} + \left(c_{D_2}-c_{P_2}\right)\right]\frac{c_\mu^{\frac{1}{2}}\sigma_{\phi_T}}{q^2 Pr_{t,l}\kappa_T^{*2}}$$
$$= \tau^+ + \frac{2q(p+q-1)-q}{q^2}y^+\frac{\partial\tau^+}{\partial y^+} + \left(\frac{p+q}{q}\right)^2$$
$$\times\frac{y^{+2}}{\tau^+}\left(\frac{\partial\tau^+}{\partial y^+}\right)^2 - \frac{p+q}{q^2}y^{+2}\frac{\partial^2\tau^+}{\partial y^{+2}} \tag{62}$$

where $Pr_{t,l}$ is the value of the turbulent Prandtl number in the logarithmic region, assumed to be 0.85. If one substitutes $\tau^+ = 1+\beta y^+$ into Eq. (62), the equation yields

$$\left[\left(c_{D_1}-c_{P_1}\right)\frac{1}{R} + \left(c_{D_2}-c_{P_2}\right)\right]\frac{c_\mu^{\frac{1}{2}}\sigma_{\phi_T}}{q^2 Pr_{t,l}\kappa_T^{*2}}$$
$$= 1 + \left[\frac{2q(p+q-1)-q}{q^2}+1\right]y^+\beta + \left(\frac{p+q}{q}\right)^2\frac{\beta^2y^{+2}}{1+\beta y^+} \tag{63}$$

For the k_T–ϵ_T and the k_T–ω_T models, respectively, Eq. (63) can be reduced to

$$\left[\left(c_{D_1}-c_{P_1}\right)\frac{1}{R} + \left(c_{D_2}-c_{P_2}\right)\right]\frac{c_\mu^{\frac{1}{2}}\sigma_{\phi_T}}{Pr_{t,l}\kappa_T^{*2}}$$
$$= \begin{cases} 1+\beta^2y^{+2}/(1+\beta y^+), & k_T\text{–}\epsilon_T \text{ model} \\ 1-2\beta y^+ & k_T\text{–}\omega_T \text{ model}\end{cases} \tag{64}$$

Considering only the first-order term, one may show that the effective von Kármán constant for temperature, κ_T^*, is modified by the dimensionless stress τ^+ according to

$$\kappa_T^* \approx \frac{\kappa_T}{\sqrt{1+\beta^2y^{+2}/(1+\beta y^+)}} \approx \kappa_T \tag{65}$$

and

$$\kappa_T^* \approx \frac{\kappa_T}{\sqrt{1-2\beta y^+}} \approx \kappa_T(1+\beta y^+) \approx \kappa_T\tau^+ \tag{66}$$

for the k_T–ϵ_T and k_T–ω_T models, respectively. As can be seen from Eqs. (66) and (65), the k_T–ω_T model follows the modified law of the wall for temperature in pressure gradient Eq. (27) whereas the k_T–ϵ_T model continues to predict the ZPG form.

The ratio of scalar to velocity time scale, R, can be shown to be equal to $[c_\mu/(c_\alpha Pr_{t,l})]^2$ in the logarithmic region. Beguier et al.[24] suggested $R \approx 0.5$ by analyzing a number of free shear flows, and it was used by many others to fix the model constant c_α; in the present work, $c_\alpha = 0.15$ is assumed. The values of c_{ϵ,P_1}, c_{ϵ,P_2}, c_{ϵ,D_1}, and c_{ϵ,D_2} were chosen to satisfy several experimental constraints and were assumed to be 0.9, 0.6, 1, and 0.8, respectively.[25] The value of σ_{k_T} is assumed to be 1, and σ_{ϕ_T} has to satisfy the law-of-the wall formula for ZPG flows, so that

$$\left[\left(c_{D_1}-c_{P_1}\right)\frac{1}{R} + \left(c_{D_2}-c_{P_2}\right)\right]\frac{c_\mu^{\frac{1}{2}}\sigma_{\phi_T}}{q^2 Pr_{t,l}\kappa_T^2} = 1 \tag{67}$$

Analogous to the example shown at the end of the previous section, one can get a family of solutions based on the choice of p and q to get the desired temperature law of the wall in ZPG [Eq. (10)] at large y^+ by integrating Eqs. (18), (19), (38) (39), (58), and (59) subject to the following boundary conditions at the wall:

$$y^+ = 0, \qquad u^+ = 0, \qquad k^+ = 0$$
$$\frac{\partial k^+}{\partial y^+} = 0, \qquad k_T^+ = 0, \qquad \frac{\partial k_T^+}{\partial y^+} = 0 \tag{68}$$

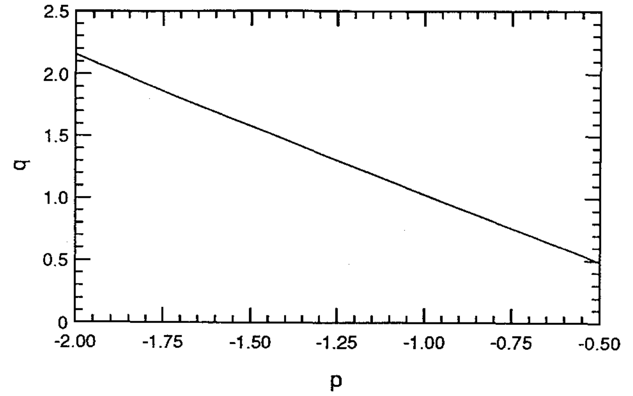


Fig. 7 Heat transfer models requiring no low-Reynolds-number terms.

If m and n are taken as -1 and 1, respectively, the curve of p and q that gives rise to the correct intercept C_T is given in Fig. 7. Again, the curve passes very close to $p = -1$, $q = 1$ (the ω_T variable).

Very close to the wall, where molecular diffusion is balanced by dissipation, the ω and ω_T equations can be simplified to

$$\frac{\partial^2 \omega^+}{\partial y^{+2}} = c_2 \omega^{+2} \tag{69}$$

$$\frac{\partial^2 \omega_T^+}{\partial y^{+2}} = c_{D_1} Pr \omega_T^{+2} + c_{D_2} Pr \omega_T^+ \omega^+ \tag{70}$$

respectively. The analytic solutions to Eqs. (69) and (70) are

$$\omega^+ = \frac{6}{c_2 y^{+2}} \tag{71}$$

$$\omega_T^+ = \left(1 - \frac{c_{D_2} Pr}{c_2}\right) \frac{6}{c_{D_1} Pr\, y^{+2}} \tag{72}$$

The limiting value of the turbulent Prandtl number at the wall can thus be expressed as

$$Pr_{t,w} = \frac{\mu_t^+}{\alpha_t^+} = \frac{c_\mu}{c_\alpha}\sqrt{\frac{\omega_T}{\omega}} = \frac{c_\mu}{c_\alpha}\sqrt{\frac{c_2 - c_{D_2} Pr}{c_{D_1} Pr}} \tag{73}$$

According to DNS results for air, $Pr_{t,w}$ is about 1.1,[11,12] and this enables a relation between c_{D_1} and c_{D_2} to be established. Moreover, Eq. (72) shows that $c_{D_1} = 0$ [implying $c_{\epsilon,D_1} = 1$, as recommended by Launder[25] from Eq. (61)] is not an acceptable choice for it appears in the denominator of the equation. Here, we have chosen $c_{D_1} = 0.1$ (hence, $c_{\epsilon,D_1} = 1.1$) and the corresponding c_{D_2} is 0.83, values close to those of Elghobashi and Launder.[20] By fixing $c_{\epsilon,P_1} = 0.9$ we may reoptimize c_{P_2} against the law of the wall of intercept, C_T. The value of c_{P_2} is found to be 0.94. It should be noted that the results presented in Fig. 7 are not much affected by the new set of constants.

Figure 8 shows calculations of Blackwell's ZPG and APG cases using the k–ω–k_T–ω_T model. In the ZPG case, the temperature profile seems to be good but the rise of the profile in the buffer layer (y^+ between 10 and 30) seems too gradual. The problem is also observed in the velocity profile. As may be seen from the figure, both velocity and temperature profiles rise to the log-law region only at $y^+ \approx 150$, as opposed to the experimental value of approximately 30. The turbulent Prandtl number seems to follow the DNS data very well for y^+ less than 10 but fails to show a fall to 0.7 in the wake region. Although the Prandtl number drops sharply only very near the edge of the boundary layer, it must be noted that its shape in this region is very much affected by the initial conditions specified in the free stream (a problem with the basic k–ω model also). In the APG case, the velocity seems to follow very well the ZPG results in the inner layer, correctly showing insensitivity of the k–ω model to the pressure gradient. The predicted temperature gradient $\partial T^+/\partial y^+$ decreases in APG and seems to follow the data well. Because the turbulent Prandtl number near the wall is not a function of pressure gradient, as seen from Eq. (73), it does not decrease in the viscous/conductive wall region as in Blackwell's data, and therefore the predicted C_T is slightly larger than the experimental value. A way to model the effect of pressure gradient on Pr_t near the wall may be required. In any event, the present calculation shows the sensitivity of the k_T–ω_T model to the pressure gradient, and its ability to increase κ_T under APG is demonstrated.

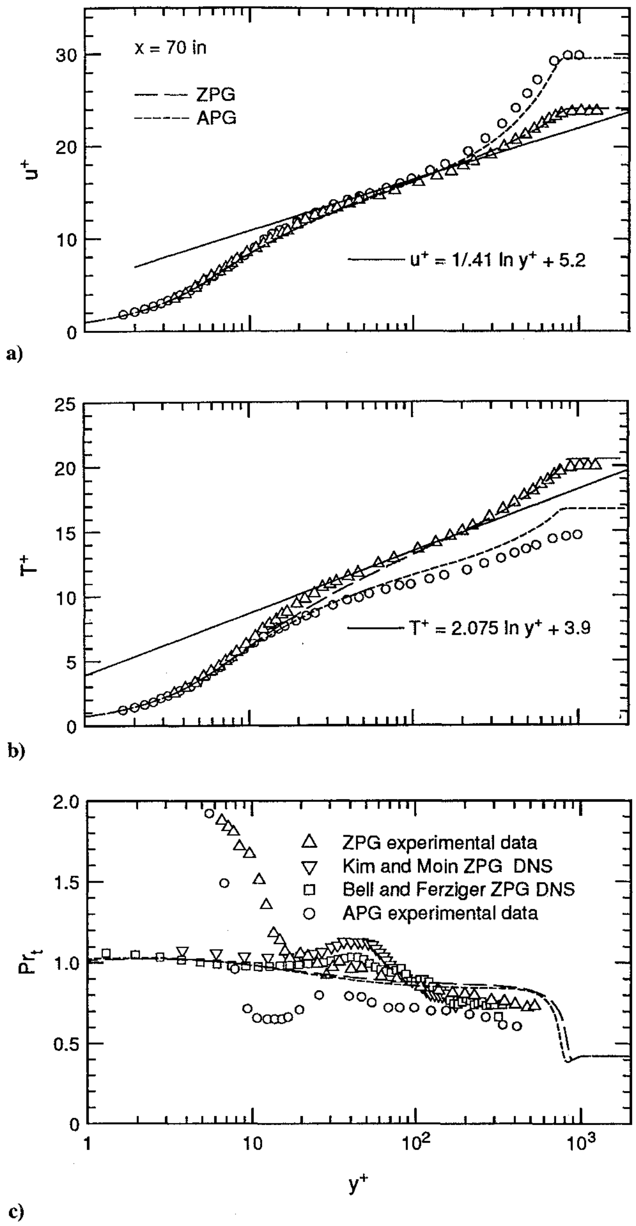


Fig. 8 Prediction of Blackwell's experiment based on k–ω–k_T–ω_T model.

V. Conclusions

Experimental evidence shows that the logarithmic law of the wall for velocity is still approximately valid in boundary layers in pressure gradient, except at low Reynolds numbers, whereas the law of the wall for temperature is strongly affected by pressure gradient (and therefore does not deserve its name). Turbulent Prandtl number in the log-law region seems to be a constant, about 0.85, independent of (moderate) pressure gradient. The mixing-length formula, which is an extension of classical wall-layer similarity arguments in which the turbulent-length scale is still assumed proportional to the distance from the wall but the velocity scale is derived from the local shear stress instead of the wall shear stress, does not correctly predict the law of the wall for velocity in pressure gradients. It follows that the mixing-length formula for heat transfer does not yield the correct temperature profile, even though the turbulent Prandtl number remains constant as extended wall-layer similarity predicts.

Near the wall, the turbulent Prandtl number rises above unity in the ZPG case. The limiting value of the turbulent Prandtl number at the wall is still controversial, but it is found that its profile for y^+ less than 10 is not critical for mean temperature prediction. In the presence of pressure gradient, it is found that the intercept of the law of the wall for temperature decreases in APG, and conversely for favorable pressure gradient, indicating that the value of the turbulent Prandtl number near the wall in APG should be less than that in ZPG, and conversely.

In both two-equation and Reynold-stress transport models, it is found that success in predicting boundary-layer flows with pressure gradients is largely dependent on the choice of the variable used in the length-scale equation. The popular ϵ variable has been found to give rise to an incorrect velocity slope in the log-law region in the presence of pressure gradient: Overall, the ω ($\equiv \epsilon/k$) variable seems to be the best choice among the family of variables, $k^m\epsilon^n$, though there is no simple physical reason for this.

Analogous to any velocity-field model, one may derive a heat transfer model and examine its prediction of the law of the wall for temperature. It is again found that the choice of the variable used in the scalar length scale equation is crucial in predicting boundary-layer flows with the pressure gradients. Overall, the $\omega_T (\equiv \epsilon_T/k_T)$ variable is again the best choice.

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